ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A NONLINEAR ELLIPTIC BOUNDARY PROBLEM

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ABSTRACT. We consider solutions to elliptic linear equation (1) of second order in an unbounded domain Q in $\mathbf{R}^n$ supposing that Q is contained in the cone

$$K = \{x = (x', x_n) \ : |x'| < Ax_n + B, \ \ 0 < x_n < \infty\},$$

and contains the cylinder

$$C = \{x = (x', x_n) \ : |x'| < 1, \ 0 < x_n < \infty\}.$$

We study the asymptotic behavior as $x_n \to \infty$ of the solutions of (1) satisfying nonlinear boundary condition (2). In dependence on the structure of Q, we obtain more precise results. In general we assume that Q is contained in the domain

$$\{x = (x', x_n) \ : |x'| < \gamma(x_n), \ \ 0 < x_n < \infty\},$$

where $1 \le \gamma(t) \le At + B$. We show that any solution of the problem growing moderately as $x_n \to \infty$, is bounded and tending to 0 as $x_n \to \infty$. In our notes [2] ,[3] we showed such a theorem for the case $\gamma(x_n) = B$, i.e. for a cylindrical domain $Q = \Omega \times (0, \infty)$, $\Omega \subset \mathbf{R}^{n-1}$.

1. Introduction. We study the solutions to the elliptic second order linear equation

$$Lu := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) - c(x)u = 0 \tag{1}$$

in an unbounded domain Q in $\mathbf{R}^n$ supposing that Q is contained in the cone

$$K = \{x = (x', x_n) \ : |x'| < Ax_n + B, \ \ 0 < x_n < \infty\},$$

and contains the cylinder

$$C = \{x = (x', x_n) \ : |x'| < 1, \ 0 < x_n < \infty\}.$$

In dependence on the structure of Q, we obtain more precise results. In general we assume that Q is contained in the domain

$$\{x = (x', x_n) \ : |x'| < \gamma(x_n), \ \ 0 < x_n < \infty\},$$

where $1 \le \gamma(t) \le At + B$, and that u satisfies the boundary condition

$$\frac{\partial u}{\partial N} + b(x)|u(x)|^{p-1}u(x) \ge 0, \quad u(x)\frac{\partial u}{\partial N} \le 0 \tag{2}$$

on the lateral surface

$$S = \{x \in \partial Q, \ \ 0 < x_n < \infty\},$$

where $p > 0$, $b(x) \ge b_0 > 0$, $\gamma \in C^1(0, \infty)$ and

$$\frac{\partial u}{\partial N} = \sum_{i=1}^{n} a_{ij}(x) \, \frac{\partial u}{\partial x_j} \, \cos\theta_i,$$

θ_i is the angle between the axis x_i and the outer normal vector.

Suppose that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2, c_0 > 0, \quad x \in Q,$$

and that $0 \leq c(x)$, $|a_{ij}(x)| \leq C$ for $i,j = 1,...,n$ and for all $x \in Q$. We don't assume that $a_{ij}(x), c(x)$ are continuous.

Let us denote Ω_T and Σ_T the sections of the domain Q and the boundary S by the plane $x_n = T$, and Q_T and S_T the parts of Q and S between the planes $x_n = 1$ and $x_n = T$.

We consider weak solutions u satisfying (1) and (2). It means that $u \in H^1_{loc}(Q) \cap L_{p+1,loc}(S)$ and

$$\int_Q [\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \psi}{\partial x_i} + c(x)u\psi]dx = 0,$$

$$\int_Q \sum_{i,j=1}^{n} [a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial u\varphi}{\partial x_i} + c(x)u(x)^2\varphi(x)]dx \leq 0,$$

$$\int_Q \sum_{i,j=1}^{n} [a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + c(x)u(x)\varphi(x)]dx + \int_S b(x)|u(x)|^{p-1}u(x)\varphi(x)dS \geq 0 \qquad (3)$$

for all functions $\psi \in H^1_0(Q)$, and positive functions $\varphi(x) \in H^1(Q)$ vanishing at $x_n = 0$ and in a neighborhood of $x_n = \infty$.

We will show that any solution of our problem growing moderately at infinity is bounded and tending to 0 as $x_n \to \infty$. In our notes [2] ,[3] we showed such a theorem for the case $\gamma(x_n) = B$, i.e. for a cylindrical domain $Q = \Omega \times (0,\infty)$, $\Omega \subset \mathbf{R}^{n-1}$.

2. Auxiliary results.

LEMMA 1. (Maximum principle). *Let D be a bounded subdomain in Q, u be a function from $H^1(Q)$, satisfying the inequality $Lu \geq 0$ in D weakly, i.e.*

$$\int_Q [\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + c(x)u\varphi]dx \leq 0$$

for all positive functions φ from $H^1_0(D)$. If $u \geq 0$ on the boundary of D, then $u \geq 0$ in D.

Proof. Let $\varphi = \min(u(x), 0)$. Then $\varphi \in H^1_0(D)$ and

$$\int_Q [\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial \varphi}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + c(x)u\varphi]dx \leq 0.$$

Therefore, *grad* $\varphi(x) = 0$ in D and thus $\varphi(x) = 0$ in D, i .e. $u \geq 0$. □

LEMMA 2. *Let K_R be a ball of radius R and $Lu = 0$ in K_R. Then*

$$\sup_{K_{R/2}} |u(x)|^2 \leq \frac{C}{R^n}\int_{K_R} |u(x)|^2 dx$$

and the constant C does not depend on R.

Proof. See [1], Theorem 5.1, p. 217 for the case $R = 1$. In the general case we use the homothety $x = Ry$. □

LEMMA 3. *Let K_R be a ball of radius R with its center at the point 0 and K_R^+ be its part situated in the domain $x_1 > 0$, let $Lu = 0$ in K_R^+ and $u\, \partial u/\partial N \le 0$ as $x_1 = 0$. Then*

$$\sup_{K_{R/2}^+} |u(x)|^2 \le \frac{C}{R^n} \int_{K_R^+} |u(x)|^2 dx$$

and the constant C does not depend on R.

Proof. The proof follows the same scheme as that of Theorem 5.1 in [1], pp. 217-223. This proof uses as its base the following inequality (5.10) from this work :

$$\int_\Omega \alpha^2 v_x^2 dx \le K \int_\Omega (\alpha^2 + \alpha_x^2) v^2 dx, \ \alpha \in C_0^1(\Omega),$$

which holds in $\Omega = K_R^+$ because of the boundary conditions $u\, \partial u/\partial N \le 0$ and can be proved by substitution of the function $\varphi = \alpha^2 u$ in the definition (3) of weak solutions. □

LEMMA 4. *Let $Lu = 0$ in Q, $u\, \partial u/\partial N \le 0$ on S. If $0 < p < 1$, we assume that $|u| \le M$. If*

$$\int_Q |\nabla u|^2 x_n^{2-n} dx + \int_S |u|^{p+1} x_n^{2-n} dS < \infty,$$

then $u(x) \to 0$ as $x_n \to \infty$ uniformly in Q.

Proof. Let $x_0 \in \Gamma$, $x_{0n} = T$, $R = \min(\gamma(T), T/2)$ and K be a ball centered at x_0 of radius R. Let $K_1 = K \cap Q$, $S_1 = K \cap S$ and K_2 be a ball of radius $R/2$, concentric with K.

If $p \le 1$, then $|u|^2 \le M^{1-p}|u|^{p+1}$ and therefore,

$$\int_S |u|^2 x_n^{2-n} dS \le C.$$

If $p > 1$, then

$$x_n u^2 \le x_n^2 |u|^{p+1} + x_n^{1-2/(p-1)}$$

and therefore,

$$\int_S |u|^2 x_n^{1-n} dS \le \int_S |u|^{p+1} x_n^{2-n} dS + \int_S x_n^{-n+1-2/(p-1)} dS \le C.$$

By the Sobolev inequality

$$\int_Q |u|^2 x_n^{-n} dx \le C_1 (\int_Q |\nabla u|^2 x_n^{2-n} dx + \int_S |u|^2 x_n^{1-n} dS) \le C_2.$$

It is clear that for any $\varepsilon > 0$ there exists a N such that

$$\int_K |u|^2 x_n^{-n} dx < \varepsilon,$$

if $T > N$, and therefore,

$$R^{-n} \int_K |u|^2 dx < C_3 \varepsilon.$$

If $S_1 = \emptyset$, then our statement follows from Lemma 2, and if $S_1 \ne \emptyset$, then it can be obtained from Lemma 3 using a partition of the unity. □

3. Conical domains. Consider firstly conical domains corresponding to the function $\gamma(x_n) = Ax_n + B$.

THEOREM 1. *Let $\gamma(x_n) = Ax_n + B$, $p > 1$. There exist constants a_0, $A_0 > 0$ such that a function u, satisfying (1) and (2) and the inequality $|u(x)| \le bx_n^a$ in the domain $Q = \{x \in \mathbf{R}^n,\ |x'| < \gamma(x_n),\ 0 < x_n < \infty\}$, with some constants $b > 0$, $0 < a < a_0$, $0 < A < A_0$, tends to 0 as $x_n \to \infty$ uniformly in Q.*

Proof. Let $h(x_n)$ be a smooth function such that $h(x_n) = 1$ as $1 < x_n < T$, $h(x_n) = 0$ for $x_n > 3T/2$ and for $x_n < 1/2$. We can assume that $|h'(x_n)| \leq C/T$ and $|h''(x_n)| \leq C/T^2$ as $x_n > T$. Set

$$J(T) = \int_Q [\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + c(x)u^2] h(x_n) x_n^{2-n} dx$$

$$+ \int_S h(x_n) b(x) |u(x)|^{p+1} x_n^{2-n} dS.$$

Substituting in (3) the function $\varphi(x) = h(x_n)x_n^{2-n}u(x)$, we obtain that

$$J(T) \leq - \int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial (x_n^{2-n} h(x_n))}{\partial x_i} u(x) dx$$

$$\leq C_1 + C_2 J(T)^{1/2} (\int_{Q_{3T/2}} |x|^{-n} u(x)^2 dx)^{1/2}.$$

Therefore, if $|u(x)| \leq b x_n^a$, then $J(T) \leq C_3 T^{2a}$.

Set now

$$I(T) = \int_{Q_T} [\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + c(x)u^2] x_n^{2-n} dx + \int_{S_T} b(x) |u(x)|^{p+1} x_n^{2-n} dS.$$

If $|u(x)| \leq b x_n^a$, then $I(T) \leq J(T) \leq C_4 T^{2a}$. Substituting in (3) the function $\varphi(x) = u(x) x_n^{2-n} h_\varepsilon(x_n)$, where $h_\varepsilon(x_n)$ is a smooth function, equal to 1 as $x_n < T - \varepsilon$ and 0 for $x_n > T$, and passing to the limit as $\varepsilon \to 0$, we obtain that

$$I(T) \leq (n-2) \int_{Q_T} \sum_{i=1}^n a_{in}(x) u \frac{\partial u}{\partial x_i} x_n^{1-n} dx + \int_{\Omega_T} \sum_{i=1}^n a_{in}(x) \frac{\partial u}{\partial x_i} u x_n^{2-n} dx' + c_1,$$

where

$$c_1 = - \int_{\Omega_1} \sum_{i=1}^n a_{in}(x) \frac{\partial u}{\partial x_i} u dx'.$$

We have

$$(n-2) \int_{Q_T} \sum_{i=1}^n a_{in}(x) u \frac{\partial u}{\partial x_i} x_n^{1-n} dx \leq C_5 (\int_{Q_T} u^2 x_n^{-n} dx)^{1/2} (\int_{Q_T} |\nabla u|^2 x_n^{2-n} dx)^{1/2};$$

$$\int_{Q_T} u(x)^2 x_n^{-n} dx \leq C_6 (A_0^2 \int_{Q_T} |\nabla u(x)|^2 x_n^{2-n} dx + \int_{S_T} u(x)^2 x_n^{1-n} dS)$$

$$\leq C_6 [A_0^2 \int_{Q_T} |\nabla u(x)|^2 x_n^{2-n} dx + \int_{S_T} (A_0^2 |u(x)|^{p+1} + A_0^{-4/(p-1)} x_n^{-1-2/(p-1)}) x_n^{2-n} dS]$$

$$\leq C_6 A_0^2 I(T) + C_7 T^{-2/(p-1)},$$

since

$$u(x)^2 \leq A_0^2 x_n |u(x)|^{p+1} + A_0^{-4/(p-1)} x_n^{-2/(p-1)}$$

as $x_n > 0$. Using the Sobolev inequality we obtain

$$\int_{\Omega_T} u(x)^2 x_n^{2-n} dx' \leq C_8 [A_0^2 T^2 \int_{\Omega_T} |\nabla u(x)|^2 x_n^{2-n} dx' + T \int_{\Sigma_T} u(x)^2 x_n^{2-n} dS]$$

$$\leq C_9 [A_0^2 T^2 \int_{\Omega_T} |\nabla u(x)|^2 x_n^{2-n} dx' + T \int_{\Sigma_T} (T |u(x)|^{p+1} + T^{-2/(p-1)}) x_n^{2-n} dS],$$

since

$$u(x)^2 \leq T|u(x)|^{p+1} + T^{-2/(p-1)}$$

as $T > 0$. If A_0 is so small that $C_9 A_0^2 \leq 1/2$, then

$$\int_{\Omega_T} u(x)^2 x_n^{2-n} dx' \leq C_{10}[T^2 I'(T) + T^{(p-3)/(p-1)}],$$

and therefore

$$\int_{\Omega_T} \sum_{i=1}^{n} a_{in}(x) \frac{\partial u}{\partial x_j} u x_n^{2-n} dx' \leq C_{11} T I'(T) + C_{12}.$$

Thus

$$I(T) \leq C_{11} T I'(T) + C_{13}.$$

Integrating this inequality we see that either $I(T) \leq C_{13}$, or $I(T) \geq C_{13} + C_{14} T^{1/C_{11}}$. Since the latter is impossible when $2aC_{11} < 1$, we obtain that $I(T) \leq C_{13}$, i.e.

$$\int_Q |\nabla u|^2 dx + \int_S |u|^{p+1} dx < \infty.$$

Now our statement follows from Lemma 4. □

EXAMPLE 1. If $Q = \{(x_1, x_2) : 0 < x_1 < x_2 < \infty\}$ and $u(x_1, x_2) = r^4 \cos 4\varphi + r^{16} \sin 16\varphi$, where r, φ are polar coordinates, then we have an example of a harmonic function of power growth in a cone. It is easy to see that $\frac{\partial u}{\partial N} + 16u|u|^3 = 0$ if $\varphi = 0$ or $\varphi = \pi/4$, so that (2) holds with $p = 4$.

4. Cylindrical domains, $p > 1$. Let now $Q = \Omega \times (0, \infty)$, where Ω is a domain in $\mathbf{R}^{n-1}$, i.e. Q is a cylinder in $\mathbf{R}^n$.

THEOREM 2. *Let $p \geq 1$. Suppose that the coefficients $c(x), a_{ij}(x)$ for $i, j = 1, \ldots, n-1$ do not depend on x_n. Let λ^2 be the first eigenvalue of the Dirichlet problem in Ω, i.e.*

$$\lambda^2 = \inf_{w \in C_0^\infty(\Omega)} \frac{\int_\Omega [\sum_{i,j=1}^{n-1} a_{ij}(x) \partial w(x)/\partial x_j \partial w(x)/\partial x_i + c(x) w(x)^2] dx}{\int_\Omega w(x)^2 dx}.$$

Let $\gamma_1 > 0, \gamma_2 > 0$ be such constants that

$$\gamma_1^2 |\sum_{j=1}^{n} a_{nj}(x) \xi_j|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j, \quad \gamma_2^2 \sum_{i,j=1}^{n-1} a_{ij}(x) \xi_i \xi_j \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j,$$

for all $\xi_j \in \mathbf{R}$. Set $\rho = \gamma_1 \gamma_2$. For any $\varepsilon \in]0, \varepsilon_0[$ there exists a positive constant a such that if $|u(x)| \leq a e^{\rho(\lambda - \varepsilon) x_n / 2}$, then $u(x) \to 0$ as $x_n \to \infty$ uniformly in Q.

Proof. Let $Q_T = \Omega \times [0, T]$, $S_T = \Gamma \times [0, T]$. Set

$$I(T) = \int_{Q_T} [\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \frac{\partial u(x)}{\partial x_i} + c(x) u^2] dx + \int_{S_T} b(x) |u(x)|^{p+1} dS.$$

Let $h \in C^\infty(\mathbf{R}), h(x_n) = 0$ for $x_n \geq T + 1$, $h(x_n) = 1$ for $0 \leq x_n \leq T$.

Putting in (3) the function $\varphi(x) = h(x_n) u(x)$, we obtain that

$$\int_Q h(x_n) \sum_{i,j=1}^{n} [a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \frac{\partial u(x)}{\partial x_i} + c(x) u^2] dx + \int_S h(x_n) b(x) |u(x)|^{p+1} dS$$

$$\leq \int_Q h'(x_n) u(x) \sum_{j=1}^{n} a_{nj}(x) \frac{\partial u(x)}{\partial x_j} dx - \int_\Omega u(x', 0) \sum_{j=1}^{n} a_{nj}(x) \frac{\partial u(x', 0)}{\partial x_j} dx'.$$

We have

$$|\int_Q h'(x_n)u(x)\sum_{j=1}^n a_{nj}(x)\frac{\partial u(x)}{\partial x_j}dx|^2 \le CI(T)\int_{Q_{T+1}\setminus Q_T} u^2(x)dx.$$

If $|u(x)| \le ae^{bx_n}$, then

$$|\int_Q h'(x_n)u(x)\sum_{j=1}^n a_{nj}(x)\frac{\partial u(x)}{\partial x_j}dx|^2 \le C_1 I(T)e^{2bT}$$

and therefore,

$$I(T) \le C_2 e^{2bT}.$$

Putting in (3) the function $\varphi(x) = u(x)h_\varepsilon(x_n)$ where $h_\varepsilon(x_n)$ is a smooth function, equal to 1 as $x_n < T - \varepsilon$ and 0 for $x_n > T$, and passing to the limit as $\varepsilon \to 0$, we obtain that

$$0 \ge \int_{Q_T}[\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial u(x)}{\partial x_i} + c(x)u^2]dx + \int_{S_T} b(x)|u(x)|^{p+1}dS$$

$$+\int_\Omega u(x',T)\sum_{j=1}^n a_{nj}(x)\frac{\partial u(x',T)}{\partial x_j}dx' - \int_\Omega u(x',0)\sum_{j=1}^n a_{nj}(x)\frac{\partial u(x',0)}{\partial x_j}dx'.$$

Let

$$\lambda_\tau^2 = \inf_{w\in C^\infty(\Omega)} \frac{\int_\Omega[\sum_{i,j=1}^{n-1} a_{ij}(x)\partial w(x)/\partial x_j \partial w(x)/\partial x_i + c(x)w^2]dx' + \tau\int_\Gamma w^2 dS}{\int_\Omega w(x)^2 dx}.$$

Since $\lambda_\tau \to \lambda$ as $\tau \to \infty$, we can choose τ so large that $\lambda_\tau \ge \lambda - \varepsilon/3$. Let C_3 be a constant, depending on τ such that for every real μ and $x \in \Omega$

$$\mu^2 \le \frac{b(x)}{\tau}|\mu|^{p+1} + C_3.$$

By the definition of the number λ, we have

$$\int_\Omega u^2(x)dx' \le \frac{1}{(\lambda - \varepsilon/3)^2}(\int_\Omega[\sum_{i,j=1}^{n-1} a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial u(x)}{\partial x_i} + c(x)u^2]dx' + \tau\int_\Gamma u^2 dS)$$

$$\le \frac{1}{\gamma_2^2(\lambda-\varepsilon/3)^2}(\int_\Omega[\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\frac{\partial u(x)}{\partial x_i} + c(x)u^2]dx'$$

$$+\int_\Gamma b(x)|u(x)|^{p+1}dS + C_3\tau).$$

On the other hand,

$$|\int_\Omega u(x',T)\sum_{j=1}^n a_{nj}(x)\frac{\partial u(x',T)}{\partial x_j}dx'|$$

$$\le (\int_\Omega u(x',T)^2 dx')^{1/2}(\int_\Omega |\sum_{j=1}^n a_{nj}(x)\frac{\partial u(x',T)}{\partial x_j}|^2 dx')^{1/2}$$

$$\le (1/\gamma_1)(\int_\Omega u(x',T)^2 dx')^{1/2} I'(T)^{1/2}.$$

Using the obtained estimates we see that

$$I(T) \le c + \frac{1}{\rho(\lambda - 2\varepsilon/3)}I'(T),$$

where

$$c = |\int_\Omega u(x,0)\sum_{j=1}^n a_{nj}(x)\frac{\partial u(x',0)}{\partial x_j}dx'| + C_4$$

and the constant C_4 depends on τ, ε, but does not depend on T.

The function $I(T)$ is increasing. If $I(T_1) \geq c$ for some $T_1 > 0$, then for $T > T_1$

$$\gamma(\lambda - 2\varepsilon/3) \leq \frac{I'(T)}{I(T) - c}.$$

It follows that $I(T) \geq c + C_5 e^{\rho(\lambda - 2\varepsilon/3)T}$. But if $|u(x)| \leq ae^{bx_n}$ and $b = \rho(\lambda - \varepsilon/2)$, then $I(T) \leq C_2 e^{2bT}$, as we have shown before. Since this is impossible for large T, we see that $I(T) \leq c$, i.e.

$$\int_Q |\nabla u|^2 dx + \int_S |u|^{p+1} dx < \infty.$$

Now we may apply Lemma 4, and the proof is complete. □

Remark 1. If the coefficients $a_{ij}(x)$ are depending on x_n, then the Theorem is true in the following sense : "There exist constants a, b such that if a solution to the problem (1), (2) satisfies the inequality

$$|u(x)| \leq ae^{bx_n},$$

then $u(x) \to 0$ as $x_n \to \infty$."

EXAMPLE 2. The following example shows that our hypothese on the growth of solutions is essential. Let $n = 2$ and $Q = \{(x_1, x_2) : -2\pi < x_1 < 2\pi, 0 < x_2 < \infty\}$, $u(x_1, x_2) = e^{x_2}\sin x_1 - e^{x_2/4}\sin(x_1/4)$. It is easy to see that the function u is harmonic in Q and $\partial u/\partial\nu + u|u|^3 = 0$ for $x_1 = \pm 2\pi$, so that (2) holds with $p = 4$.

5. Other domains, $p > 1$. Let now $|x'| \leq \gamma(x_n), 0 < x_n < \infty, x = (x', x_n) \in Q$. Let

$$F(t) = \int_0^t \frac{ds}{\gamma(s)}.$$

Suppose that $\gamma(s) = o(s)$ as $s \to \infty$, $\gamma(s) \to \infty$ as $s \to \infty$, and $\gamma(T + s) \leq C\gamma(T)$ if $0 < s \leq C_1\gamma(T)$. Note that $F(t) \to \infty$ as $t \to \infty$.

THEOREM 3. *If u is a solution of equation (1) in Q, satisfying (2), and*

$$|u(x)| \leq be^{aF(x_n)}$$

in Q with a small enough constant a, then $u(x) \to 0$ as $x_n \to \infty$ uniformly in Q.

Proof. Let $h(x_n)$ be a smooth function such that $h(x_n) = 1$ for $1 < x_n < T$, $h(x_n) = 0$ for $x_n > 3T/2$. We may assume that $|h'(x_n)| \leq C/T$ and $|h''(x_n)| \leq C/T^2$ if $x_n > T$. Put

$$J(T) = \int_Q [\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}h(x_n)x_n^{2-n} + c(x)u^2]dx$$

$$+ \int_S h(x_n)b(x)|u(x)|^{p+1}x_n^{2-n}dS.$$

Putting in (3) the function $\varphi(x) = h(x_n)x_n^{2-n}u(x)$, we obtain that

$$J(T) \leq \int_Q \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial(x_n^{2-n}h(x_n))}{\partial x_i}u(x)dx$$

$$\leq C_1 + C_2 J(T)^{1/2}(\int_{Q_{3T/2}} x_n^{-n}u(x)^2 dx)^{1/2}.$$

Therefore, if $|u(x)| \le be^{aF(x_n)}$, then $J(T) \le C_3 e^{2aF(T)}$.
Let now

$$I(T) = \int_{Q_T} [\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + c(x)u^2] x_n^{2-n} dx + \int_{S_T} b(x)|u(x)|^{p+1} x_n^{2-n} dS.$$

If $|u(x)| \le be^{aF(x_n)}$, then $I(T) \le J(T) \le C_4 e^{2aF(T)}$. Putting in (3) the function $\varphi(x) = u(x)x_n^{2-n}h_\varepsilon(x_n)$, where $h_\varepsilon(x_n)$ is a smooth function, equal to 1 as $x_n < T - \varepsilon$ and 0 for $x_n > T$, and passing to the limit as $\varepsilon \to 0$, we obtain that

$$I(T) \le -\int_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x) u \frac{\partial u}{\partial x_j} \frac{\partial x_n^{2-n}}{\partial x_i} dx + \int_{\Omega_T} \sum_{i=1}^{n} a_{in}(x) \frac{\partial u}{\partial x_j} u x_n^{2-n} dx' + c_1,$$

where

$$c_1 = -\int_{\Omega_1} \sum_{i=1}^{n} a_{in}(x) \frac{\partial u}{\partial x_j} u x_n^{2-n} dx'.$$

By the Sobolev inequality,

$$\int_{\Omega_T} u(x)^2 x_n^{2-n} dx' \le C_5[\gamma(T)^2 \int_{\Omega_T} |\nabla u(x)|^2 x_n^{2-n} dx' + \gamma(T) \int_{\Sigma_T} u(x)^2 x_n^{2-n} dS]$$

$$\le C_6[\gamma(T)^2 \int_{\Omega_T} |\nabla u(x)|^2 x_n^{2-n} dx' + \gamma(T) \int_{\Sigma_T} x_n^{2-n} (\gamma(T)|u(x)|^{p+1} + \gamma(T)^{-\frac{2}{p-1}}) dS],$$

i.e.

$$\int_{\Omega_T} u(x)^2 x_n^{2-n} dx' \le C_7[\gamma(T)^2 I'(T) + \gamma(T)^{\frac{p-1}{p+1}}],$$

and therefore,

$$\int_{\Omega_T} \sum_{i=1}^{n} a_{in}(x) \frac{\partial u}{\partial x_j} u x_n^{2-n} dx' \le C_8 \gamma(T) I'(T) + c.$$

On the other hand, using translation $x_n \to x_n + T$, we can suppose that $\gamma(x_n) \le \varepsilon x_n$ for $x_n \ge 0$. Using the inequality

$$u^2 x_n^{1-n} \le u^{p+1} x_n^{2-n} + x_n^{1-n-2/(p-1)},$$

we see that

$$|\int_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x) u \frac{\partial u}{\partial x_j} \frac{\partial x_n^{2-n}}{\partial x_i} dx| \le C_9 I(T)^{1/2} (\int_{Q_T} u(x)^2 x_n^{-n} dx)^{1/2}$$

$$\le C_{10} I(T)^{1/2} (\int_{Q_T} |\nabla u(x)|^2 \gamma(x_n)^2 x_n^{-n} dx + \int_{S_T} u(x)^2 \gamma(x_n) x_n^{-n} dS]^{1/2}$$

$$\varepsilon \le C_{11} I(T)^{1/2} (\int_{Q_T} |\nabla u(x)|^2 x_n^{2-n} dx + \int_{S_T} u(x)^2 x_n^{1-n} dS]^{1/2}$$

$$\le \varepsilon C_{12} I(T)^{1/2} \left(\int_{Q_T} |\nabla u(x)|^2 x_n^{2-n} dx + \int_{S_T} u(x)^{p+1} x_n^{2-n} dS + T^{-2/(p-1)} \right)^{1/2}$$

$$\le \varepsilon C_{13}(I(T) + 1).$$

If ε is so small that $C_{13}\varepsilon \le 1/2$, it follows the inequality

$$I(T) \le c + C_{14} \gamma(T) I'(T).$$

Integrating this inequality we see that either $I(T) \leq c$, or $I(T) \geq c + C_{15}e^{F(T)/C_{14}}$. Since the latter is impossible when $2aC_{14} < 1$, we obtain that $I(T) \leq c$, i.e.

$$\int_Q |\nabla u|^2 dx + \int_S |u|^{p+1} dx < \infty.$$

Applying Lemma 4, we can complete the proof. □

6. Case $0 < p < 1$**.** Let us consider now the case $0 < p < 1$.

THEOREM 4. *Let* $0 < p < 1$ *and* $0 < a < 1/(1-p)$*. Let* $\gamma(x_n) \leq Ax_n + B$ *and* $A < A_0$ *with small enough* A_0*. If* $|u(x)| \leq bx_n^a$ *with some* b*, then* $u(x) \to 0$ *as* $x_n \to \infty$ *uniformly in* Q*.*

Proof. **1.** Let $h(x_n)$ be a smooth function such that $h(x_n) = 1$ for $1 < x_n < 2T$, $h(x_n) = 0$ for $x_n < 1/2$ and for $x_n > 5T/2$. We can assume that $|h'(x_n)| \leq C/T$ and $|h''(x_n)| \leq C/T^2$, if $x_n > 2T$. Put

$$J(T) = \int_Q \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} h(x_n) x_n^{2-n} dx + \int_S h(x_n) b(x) |u(x)|^{p+1} x_n^{2-n} dS.$$

Putting in (3) the function $\varphi(x) = h(x_n) x_n^{2-n} u(x)$, we obtain that

$$J(T) = \int_Q [\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial (x_n^{2-n} h(x_n))}{\partial x_i} u(x) + c(x) u^2] dx$$

$$\leq C_1 + C_2 J(T)^{1/2} (\int_Q |x|^{-n} u(x)^2 dx)^{1/2}.$$

Therefore, if $|u(x)| \leq bx_n^a$, then $J(T) \leq C_3 T^{2a}$.

2. Put

$$I(T) = \int_{Q_T} [\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} x_n^{2-n} + c(x) u^2] dx + \int_{S_T} b(x) |u(x)|^{p+1} x_n^{2-n} dS.$$

If $|u(x)| \leq bx_n^a$, then $I(T) \leq J(T) \leq C_4 T^{2a}$. Putting in (3) the function $\varphi(x) = u(x) x_n^{2-n} h_\varepsilon(x_n)$, where $h_\varepsilon(x_n)$ is a smooth function, equal to 1 as $x_n < T - \varepsilon$ and to 0 for $x_n > T$, and passing to the limit as $\varepsilon \to 0$, we obtain that

$$I(T) \leq -\frac{1}{2} \int_{Q_T} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u^2}{\partial x_j} \frac{\partial x_n^{2-n}}{\partial x_i} dx + \int_{\Omega_T} \sum_{i=1}^n a_{in}(x) \frac{\partial u}{\partial x_j} u x_n^{2-n} dx' + c_1,$$

where

$$c_1 = -\int_{\Omega_1} \sum_{i=1}^n a_{in}(x) \frac{\partial u}{\partial x_j} u x_n^{2-n} dx'.$$

By the Sobolev inequality

$$\int_{\Omega_T} u(x)^2 x_n^{2-n} dx' \leq C_5 [T^2 \int_{\Omega_T} |\nabla u(x)|^2 x_n^{2-n} dx' + T \int_{\Sigma_T} u(x)^2 x_n^{2-n} dS]$$

$$\leq C_6 [T^2 \int_{\Omega_T} |\nabla u(x)|^2 x_n^{2-n} dx' + (bT^a)^{1-p} \int_{\Sigma_T} |u(x)|^{1+p} x_n^{2-n} dS],$$

i.e.

$$\int_{\Omega_T} u(x)^2 x_n^{2-n} dx' \leq C_7 [T^2 + (bT^a)^{1-p}] I'(T),$$

and therefore,

$$\int_{\Omega_T}\sum_{i=1}^n a_{in}(x)\frac{\partial u}{\partial x_j}ux_n^{2-n}dx' \le C_8TI'(T)+c.$$

Thus

$$I(T)\le c+C_9TI'(T).$$

Integrating this inequality we see that either $I(T)\le c$, or $I(T)\ge c+C_{10}T^{1/C_9}$. Since the latter is impossible if $2aC_9<1$, we obtain that $I(T)\le c$.

3. Let us show that the last inequality implies that u is bounded in Q. Let $M=\max_{x_n=1}|u(x))|$. Put

$$w(x)=\max(0,u(x)-M).$$

Let us substitute in (2) the function $\varphi(x)=\theta(x_n)x_n^{2-n}w(x)$. where $\theta(x_n)$ is a continuous function, linear for $T<x_n<2T$ and such that $\theta(x_n)=1$ for $0<x_n<T$, $\theta(x_n)=0$ for $x_n>2T$. Then $|\theta'(x_n)|\le T^{-1}$. Since $w(x',0)=0$, we obtain that

$$\int_Q[\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial w}{\partial x_i}+c(x)uw]x_n^{2-n}\theta(x_n)dx$$

$$+\int_S b(x)|u(x)|^{p-1}u(x)w(x)\theta(x_n)x_n^{2-n}dS$$

$$\le-\int_Q w(x)\sum_{j=1}^n a_{nj}(x)\frac{\partial u}{\partial x_j}\frac{\partial[x_n^{2-n}\theta(x_n)]}{\partial x_n}dx.$$

It is clear that

$$\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_j}=\frac{\partial w}{\partial x_{ij}}\frac{\partial w}{\partial x_j},\quad \frac{\partial u}{\partial x_j}w(x)=\frac{\partial w}{\partial x_j}w(x),$$

$$u(x)w(x)\ge w(x)^2,\quad |u(x)|^{p-1}u(x)w(x)\ge M^{p-1}w(x)^2.$$

Thus

$$I\equiv\int_Q[\sum_{i,j=1}^n a_{ij}(x)\frac{\partial w}{\partial x_j}\frac{\partial w}{\partial x_i}+c(x)w^2]\theta(x_n)x_n^{2-n}dx+\int_{S_T}b(x)|w(x)|^{p+1}\theta(x_n)x_n^{2-n}dS$$

$$\le-\int_{Q_T}w(x)\sum_{j=1}^n a_{nj}(x)\frac{\partial w}{\partial x_j}\frac{\partial[x_n^{2-n}\theta(x_n)]}{\partial x_n}dx$$

$$\le\frac{C_{10}}{T}[\left(\int_{Q_{2T}\setminus Q_T}w^2x_n^{2-n}dx\right)^{1/2}\left(\int_{Q_{2T}\setminus Q_T}|\nabla w|^2x_n^{2-n}dx\right)^{1/2}$$

$$+\left(\int_{Q_{2T}}w^2\theta(x_n)x_n^{2-n}dx\right)^{1/2}\left(\int_{Q_{2T}}|\nabla w|^2\theta(x_n)x_n^{2-n}dx\right)^{1/2}]$$

$$\le\frac{C_{11}}{T}A_0^2T^2\int_{Q_{2T}\setminus Q_T}|\nabla w|^2x_n^{2-n}dx+\frac{C_{11}}{T}[A_0^2T^2\int_{Q_{2T}}|\nabla w|^2\theta(x_n)x_n^{2-n}dx$$

$$+T\left(\int_{S_{2T}}\theta(x_n)x_n^{2-n}w(x)^2dS\right)^{1/2}\left(\int_{Q_{2T}\setminus Q_T}|\nabla w|^2\theta(x_n)x_n^{2-n}dx\right)^{1/2}]$$

$$\le C_{12}\left[\int_{Q_{2T}\setminus Q_T}|\nabla w|^2x_n^{2-n}dx+\frac{1}{T}(bT^a)^{1-p}\int_{S_{2T}\setminus S_T}x_n^{2-n}|w(x)|^{p+1}dS\right]+C_{12}A_0I.$$

Since

$$\int_{S_{2T}\setminus S_T}x_n^{2-n}dS\le C_{13}T$$

and

$$\int_{Q_{2T}\setminus Q_T} |\nabla w|^2 x_n^{2-n} dx \to 0, \int_{S_{2T}\setminus S_T} x_n^{2-n} w(x)^{p+1} dS \to 0,$$

as $T \to \infty$, and if A_0 is so small that $C_{12}A_0 < 1/2$, we have

$$\int_{Q_T} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial w}{\partial x_j}\frac{\partial w}{\partial x_i} x_n^{2-n} dx + \int_{S_T} b(x)|w(x)|^{p+1} x_n^{2-n} dS \to 0$$

if $T \to \infty$. This means that $w(x) = 0$, i.e. $u(x) \le M$. In the same way one can see that $u(x) \ge -M$. Since

$$\int_Q |\nabla u|^2 dx + \int_S |u|^{p+1} dx < \infty,$$

the proof is complete after application of Lemma 4. □

EXAMPLE 3. The function $u(x_1, x_2) = x_1^3 + 3x_1(1 - x_2^2)$ is harmonic for $x_1 > 0$ and satisfies the condition $\partial u/\partial N + 6u^{1/3} = 0$ as $x_2 = \pm 1$, so that (2) holds with $p = 1/3$. Moreover, $u(0, x_2) = 0$, $u(x_1, x_2) > 0$ as $x_1 > 0$. It is obvious that $u(x_1, x_2) \le 2x_1^3$ for $x_1 > 2$.

7. The case of positive $c(x)$. We'll show that that the structiure of the domain Q is inessential if the function c is strictly positive.

THEOREM 5. *Let*

$$Q \subset \{x = (x', x_n) \in \mathbf{R}^n, |x'| \le Ax_n + B, 0 < x_n < \infty\}.$$

Suppose that $u(x)$ *is a weak solution of* (1) *in* Q *such that* $u\frac{\partial u}{\partial N} \le 0$ *on* S, $c(x) \ge c_0 > 0$. *There exists a positive constant* a *such that if* $|u(x)| \le be^{ax_n}$, *then* $u(x) \to 0$ *as* $x_n \to \infty$ *uniformly in* Q.

Proof. Let $h(x_n)$ be a smooth function such that $h(x_n) = 1$ as $1 < x_n < T$, $h(x_n) = 0$ for $x_n > 3T/2$ and for $x_n < 1/2$. We can assume that $|h'(x_n)| \le C/T$ and $|h''(x_n)| \le C/T^2$ as $x_n > T$. Set

$$J(T) = \int_Q [\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} + c(x)u^2] h(x_n) x_n^{2-n} dx$$

$$+ \int_S h(x_n) b(x) |u(x)|^{p+1} x_n^{2-n} dS.$$

Substituting in (3) the function $\varphi(x) = h(x_n)x_n^{2-n}u(x)$, we obtain that

$$J(T) = -\int_Q \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial (x_n^{2-n} h(x_n))}{\partial x_i} u(x) dx$$

$$\le C_1 + C_2 J(T)^{1/2} (\int_{Q_{3T/2}} |x|^{-n} u(x)^2 dx)^{1/2}.$$

Therefore, if $|u(x)| \le e^{ax_n}$, then $J(T) \le C_3 e^{2aT}$.

Set now

$$I(T) = \int_{Q_T} [\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} + c(x)u^2] x_n^{2-n} dx + \int_{S_T} b(x)|u(x)|^{p+1} x_n^{2-n} dS.$$

If $|u(x)| \leq be^{ax_n}$, then $I(T) \leq J(T) \leq C_3 e^{2aT}$. Substituting in (3) the function $\varphi(x) = u(x)x_n^{2-n}h_\varepsilon(x_n)$, where $h_\varepsilon(x_n)$ is a smooth function, equal to 1 as $x_n < T - \varepsilon$ and 0 for $x_n > T$, and passing to the limit as $\varepsilon \to 0$, we obtain that

$$I(T) \leq (n-2)\int_{Q_T} \sum_{i=1}^{n} a_{in}(x) u \frac{\partial u}{\partial x_i} x_n^{1-n} dx + \int_{\Omega_T} \sum_{i=1}^{n} a_{in}(x) \frac{\partial u}{\partial x_i} u x_n^{2-n} dx' + c_1,$$

where

$$c_1 = -\int_{\Omega_1} \sum_{i=1}^{n} a_{in}(x) \frac{\partial u}{\partial x_i} u dx'.$$

We have

$$(n-2)\int_{Q_T} \sum_{i=1}^{n} a_{in}(x) u \frac{\partial u}{\partial x_i} x_n^{1-n} dx \leq C_4 \left(\int_{Q_T} u^2 x_n^{-n} dx\right)^{1/2} \left(\int_{Q_T} |\nabla u|^2 x_n^{2-n} dx\right)^{1/2}$$

$$\leq \varepsilon \int_{Q_T} |\nabla u(x)|^2 x_n^{2-n} dx + C_5/\varepsilon \int_{Q_T} |\nabla u(x)|^2 x_n^{2-n} dx.$$

Using the translation $x_n \to X_n + T_0$, and choosing ε and T_0 we obtain that

$$(n-2)\int_{Q_T} \sum_{i=1}^{n} a_{in}(x) u \frac{\partial u}{\partial x_i} x_n^{1-n} dx \leq I(T)/2.$$

On the other hand,

$$\int_{\Omega_T} u(x)^2 x_n^{2-n} dx' \leq C_5 I'(T).$$

Thus

$$I(T) \leq C_6 T I'(T) + C_7.$$

Integrating this inequality we see that either $I(T) \leq C_7$, or $I(T) \geq C_7 + C_8 T^{1/C_6}$. Since the latter is impossible when $2aC_6 < 1$, we obtain that $I(T) \leq C_7$, i.e.

$$\int_Q |\nabla u|^2 dx + \int_S |u|^{p+1} dx < \infty.$$

Now our statement follows from Lemma 4. □

8. Existence of positive solutions.

THEOREM 5. *Let*

$$Q \subset \{x = (x', x_n) \in \mathbf{R}^n,\ |x'| \leq Ax_n + B, 0 < x_n < \infty\}.$$

For any $p > 0$ there exists a function $u(x)$, positive in Q, satisfying the equation $Lu = 0$ and boundary condition

$$\frac{\partial u}{\partial N} + b(x)u(x)^p = 0 \quad \text{on} \quad S$$

and such that $u(x', 0) = 1$.

Proof. Let u_T be a solution to the equation $Lu = 0$ in Q_T, satisfying boundary condition (2) and such that

$$u_T(x', 0) = 1, \quad u_T(x', T) = 0. \tag{4}$$

Such a solution can be found by minimizing the functional

$$F_T(u) = \int_{Q_T} \left[\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \frac{\partial u(x)}{\partial x_i} + c(x)u^2\right] dx + \frac{2}{p+1}\int_{S_T} b(x)u(x)^{p+1} dS$$

in the class of positive functions u from $C^\infty(Q_T)$, satisfying condition (4). The minimizing function u_T is positive, $0 < u_T(x) < 1$ in Q_T.

Moreover,

$$\int_Q [\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u_T}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + c(x)u_T\varphi]dx + \int_S b(x)u_T(x)^p\varphi(x)dS = 0 \tag{5}$$

for all functions $\varphi(x) \in H^1(Q)$, equal to 0 for $x_n = 0$ and for $x_n = T$. The function $u_T(x)$ is continuous in Q_T, see [1].

Set $u_T(x) = 0$ in Q for $x_n > T$.

Let K be a compact subset in $\bar{Q}$, and let T_0 be such that $K \subset \bar{Q}_{T_0}$. Let $h(x_n)$ be a piece-wise function such that $h(x_n) = 1$ for $1 < x_n < T_0 - 1$, $h(x_n) = x_n$ for $0 < x_n < 1$, $h(x_n) = 0$ for $x_n > T_0$.

Substitute in (5) the function $\varphi(x) = h(x_n)u_T(x)$, where $T > T_0$. We obtain

$$J(T,T_0) \equiv \int_Q [\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u_T}{\partial x_j}\frac{\partial u_T}{\partial x_i} + c(x)u_T^2(x)]h(x_n)dx$$

$$+\int_S h(x_n)b(x)u_T(x)^{p+1}dS = -\int_Q \sum_{j=1}^n a_{nj}(x)\frac{\partial u_T}{\partial x_j}h'(x_n)u_T(x)dx$$

$$\le C_1 J(T,T_0)^{1/2}(\int_{Q_{T_0}} u_T(x)^2 dx)^{1/2}$$

$$\le C_1 J(T,T_0)^{1/2}(\int_{Q_{T_0}} dx)^{1/2} \le C(T_0)J(T,T_0)^{1/2}.$$

Therefore, $J(T,T_0) \le C_1(T_0)$.

Therefore, the set of bounded functions u_T on K is weakly compact, i.e. there exists a subsequence of positive functions $\{u_{T_k}\}$, weakly converging in $H^1(K) \cap L_{p+1}(S \cap \bar{K})$ to a function u. Choosing a sequence of compact sets K_m, tending to Q and using diagonalization, one can find a subsequence which will be denoted as $\{u_k\}$, converging everywhere in Q and in the space $H^1_{loc}(Q) \cap L_{p+1,loc}(S)$ to a function u from this space.

We have

$$\int_Q [\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u_k}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + c(x)u_k\varphi]dx + \int_S b(x)u_k(x)^p\varphi(x)dS = 0$$

for all functions $\varphi(x) \in H^1(Q)$, equal to 0 for $x_n = 0$ and for $x_n = T_k$. Passing to the limit we obtain that

$$\int_Q [\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + c(x)u\varphi]dx + \int_S b(x)u(x)^p\varphi(x)dS = 0$$

for all functions $\varphi(x) \in H^1(Q)$, equal to 0 for $x_n = 0$ and in a neighborhood of infinity, i.e. u is a weak solution to the problem (1)-(2). Moreover, $u \ge 0$ in Q and $u(x',0) = 1$, so that $u \not\equiv 0$.

The proof is complete. □

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